Supplementary material

**Feasibility of using optimal control theory and training-performance model to design optimal training programs for athletes**

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**Section 1**

In this section, we provide a brief introduction regarding how fitness, fatigue and training load (three variables in the Bannister impulse-response model) are typically measured in experiments.

It was found that some measurable physiological parameters and questionnaire scores are more or less correlated to fitness and fatigue in the Banister IR model (or models modified based on it). However, there does not exist a single measurable metric that can typically represent fitness or fatigue (even for a metric that is found to be strongly correlated to fitness or fatigue), since these two conceptual variables in the model represent the overall effects of multiple factors that contribute to the dynamical response of performance. To the best of our knowledge, in experimental studies in which the Banister IR model (or models modified based on it) is used to curve-fitted data, most researchers adopt to experimentally measure performance (using a chosen metric, e.g., time, distance, speed, power) rather than fitness and fatigue. Once a performance data set is curve-fitted by such a model, fitness and fatigue can be quantified. Please see references (Taha and Thomas, 2003; Borresen and Lambert, 2009) for reviews of physiological correlates of fitness and fatigue.

Training load is defined as the multiplication of the intensity by the time duration of the training session. There exists a number of metrics for quantifying training load, including external training load measures (defined as the amount of work an athlete undertakes, e.g., distance, speed, power) and internal training load measures (defined as the psychophysiological effects induced by the external training load on an athlete, e.g., heart rate, rating of perceived exertion). The most well-known metrics for quantifying training load could be the Banister training impulse (TRIMP) and the subsequent modified TRIMP-based metrics (Vermeire et al. 2021).

The Banister training impulse is calculated using the equation

|  |  |
| --- | --- |
|  | (S1) |

where is the time duration of the training session (in minute), is a scaling constant and for men or for women. is the fraction of the heart rate (HR) reserve that is served as a measure

of the intensity, where and are the average and maximum HR during the training session, and is the resting HR.

The Edwards TRIMP (eTRIMP) is a kind of modified TRIMP. To calculate eTRIMP, five zones are defined according to the percentage of : zone 1, 50%–59% ; zone 2, 60%–69% ; zone 3, 70%–79% ; zone 4, 80%–89% ; zone 5, 90%–100% . The equation for calculating eTRIMP is

|  |  |
| --- | --- |
|  | (S2) |

where is a scaling constant equal to 1, 2, 3, 4 and 5 for each zone, respectively. is the time duration in each zone. is the fraction of HR in each zone, where is the maximum HR in each zone.

The Lucia TRIMP (luTRIMP) is another kind of modified TRIMP. To calculate luTRIMP, three zones are defined according to gas exchange threshold (GET) and respiratory compensation point (RCP): zone 1, below GET; zone 2, between GET and RCP; zone 3, above RCP. The equation for calculating luTRIMP is

|  |  |
| --- | --- |
|  | (S3) |

where is a scaling constant equal to 1, 2 and 3 for each zone, respectively. is the time duration in each zone.

In addition to the three TRIMP-based metrics introduced above, there exists other kinds of metrics for quantifying training load such as the three introduced below.

To calculate a metric called session RPE (sRPE), the subject is asked to rate his/her RPE using the CR-10 scale after the training session, then sRPE is calculated by multiplying this RPE by the time duration of the training session.

The equation for calculating another metric called training stress score (TSS) is

|  |  |
| --- | --- |
|  | (S4) |

where is the time duration of the training session and is the normalized power. is equal to , where is the functional threshold power of the subject.

The last metric introduced here is the mechanical energy spent during the training session, calculated by multiplying the power output by the time duration of the training session.

References:

1. Taha, T., & Thomas, S. G. (2003). Systems modelling of the relationship between training and performance. Sports Medicine, 33, 1061-1073.
2. Borresen, J., & Lambert, M. I. (2009). The quantification of training load, the training response and the effect on performance. Sports Medicine, 39, 779-795.
3. Vermeire, K. M., Van de Casteele, F., Gosseries, M., Bourgois, J. G., Ghijs, M., & Boone, J. (2021). The influence of different training load quantification methods on the fitness-fatigue model. International Journal of Sports Physiology and Performance, 16(9), 1261-1269.

**Section 2**

In this section, we present a proof of the existence and uniqueness of solutions of the optimal control problem in the present article. The optimal control problem in the present article is formulated as

|  |  |
| --- | --- |
|  | (S5) |
| subject to |  |
|  | (S6) |
|  | (S7) |
|  | (S8) |
|  | (S9) |

where is the control variable (which is a piecewise continuous function), while , and are the state variables (which are continuous functions). and are the lower and upper bounds of , respectively. , , , , and are parameters. Please see the main article for understanding the physical meanings of these variables and parameters.

**PROOF**

First, we need a definition and a theorem for the proof. The definition, theorem and logic for the proof are referred to the reference (Lenhart and Workman, 2007).

**DEFINITION 1**

A function is said to be concave on if

for all and for any .

**THEOREM 1**

Consider

|  |
| --- |
|  |
| subject to |
|  |
|  |

Suppose that and are both continuously differentiable functions in their all argument and concave in and . Suppose is a control, with associated state , and a piecewise differentiable function, such that , and together satisfy on :

|  |
| --- |
|  |
|  |
|  |
|  |

Then for all control , we have

|  |
| --- |
| . |

Let’s begin the proof.

The objective functional of the optimal control problem is

|  |  |
| --- | --- |
|  | (S10) |

where

|  |  |
| --- | --- |
|  | (S11) |

Let and be any two admissible controls, and their convex combination is defined as

|  |  |
| --- | --- |
|  | (S12) |

Letting in Eq. (S11), then Eq. (S11) becomes

|  |  |
| --- | --- |
|  | (S13) |

On the other hand,

|  |  |
| --- | --- |
|  | (S14) |

Subtracting Eq. (S14) from Eq. (S13), we have

|  |  |
| --- | --- |
|  | (S15) |

Since , and are all positive, Eq. (S15) guarantees that

|  |  |
| --- | --- |
|  | (S16) |

Hence, according to **DEFINITION 1** and Eq. (S15), we have proved that in the objective functional is strictly concave.

On the other hand, since is a parameter of the problem that is a constant, and in the objective functional are constants.

Since is strictly concave and and are constants, the objective functional must be strictly concave. In addition, , and are continuously functions.

Hence, according to **THEOREM 1**, we have proved that the optimal control problem has the unique solution.

Next, we are going to apply Pontryagin's Maximum Principle to show the necessary conditions for the unique optimal control variable.

The Hamiltonian of the optimal control problem is

|  |  |
| --- | --- |
|  | (S17) |

From the Hamiltonian, the necessary conditions can be obtained as

|  |  |
| --- | --- |
|  | (S18) |

where is the optimal control variable.

|  |  |
| --- | --- |
|  | (S19) |
|  | (S20) |
|  | (S21) |

Equation (S19) can further prove that the objective functional is strictly concave.

The solutions of Eqs. (S20) and (S21) are

|  |  |
| --- | --- |
| and , respectively. | (S22) |

The optimal control variable can be obtained by Eq. (S18) as

|  |  |
| --- | --- |
|  | (S23) |

Substituting Eq. (S22) into Eq. (S23), we have

|  |  |
| --- | --- |
|  | (S24) |

Since is constrained through Eq. (S8), the optimal control variable is

|  |  |
| --- | --- |
|  | (S25) |

It is the analytical solution of the optimal control variable of the optimal control problem. If we let in Eqs. (S6) and (S7), these two differential equations theoretically define the analytical solutions of the optimal state variables and .

The existence of the analytical solutions can further help to prove that the optimal control problem has the unique solution.

Reference:

Lenhart, S., & Workman, J. T. (2007). Optimal control applied to biological models. Chapman and Hall/CRC.

**Section 3**

In the main article, we present the simulation results with , representing the individual characteristics of an athlete. The simulation results of another set of parameters , representing the individual characteristics of another athlete, are presented in this section, as below.



**Figure S1:** The simulation results without using optimal control theory.



**Figure S2:** The simulation results with using optimal control theory with .



**Figure S3:** The simulation results with using optimal control theory with .



**Figure S4:** The simulation results with using optimal control theory with .

**Table S1:** The values of the performance on the competition day (), the cumulated training load during the training course () and the efficiency coefficient () of all simulation experiments with and without using optimal control theory (OCT).

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  | w/ using OCT | | | w/o using OCT |
| Range of |  |  |  | n/a |
|  | 2080 | 2045 | 2013 | 2012 |
|  | 10300 | 10400 | 10654 | 12390 |
|  | 0.2019 | 0.1966 | 0.1889 | 0.1624 |

**Section 4**

In this section, we present the results of four more simulation experiments without using optimal control theory, and compare them to the results of a simulation experiment with using optimal control theory, as below. In each of these simulation experiments without using optimal control theory, a specifically designed training program is used for the simulation. The setting of the parameters and initial conditions of these simulation experiments is the same as that in the main article.



**Figure S5:** The results of the simulation experiment No. 1 without using optimal control theory.



**Figure S6:** The results of the simulation experiment No. 2 without using optimal control theory.



**Figure S7:** The results of the simulation experiment No. 3 without using optimal control theory.



**Figure S8:** The results of the simulation experiment No. 4 without using optimal control theory.

**Table S2:** The values of the performance on the competition day (), the cumulated training load during the training course () and the efficiency coefficient () of a simulation experiment with using optimal control theory (OCT) and four simulation experiments without using optimal control theory.

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | w/ using OCT | w/o using OCT | | | |
| Range of (for w/ using OCT)  or number of the simulation experiments (for w/o using OCT) |  | No. 1 | No. 2 | No. 3 | No. 4 |
|  | 866 | 433 | 527 | 476 | 606 |
|  | 8018 | 12800 | 6500 | 9550 | 11150 |
|  | 0.1080 | 0.034 | 0.0811 | 0.0498 | 0.0544 |